Int. J. Solids Structures Vol. 28, No. 8, pp. 1041–1052, 1991
Printed in Great Britain.

0020–7683/91 $3.00 + .00

A MODEL FOR LOAD-TRANSFER FROM AN EMBEDDED FIBER TO AN ELASTIC MATRIX

WILLIAM S. SLAUGHTER and J. LYELL SANDERS, JR
Division of Applied Sciences, Harvard University, Cambridge, MA 02138, U.S.A.

(*Received* 21 *August* 1990; *in revised form* 2 *December* 1990)

Abstract—A model is presented for approximating load-diffusion from axially loaded fibers embedded in elastic matrices. The fundamental elastostatic solutions used are for a point force and a point dilatation in either a fully-infinite or semi-infinite space. Tangential tractions across the fiber–matrix interface are included explicitly in the analysis. The model is applied to the three-dimensional analogs of Melan's first problem and Reissner's problem and comparisons are made with exact results in the case of the former to help establish the validity of the model.

1. INTRODUCTION

The ability to analyze load-transfer in fiber–matrix systems which are illustrative of those that exist in fiber-reinforced materials is fundamental to the study of how such materials behave in application. Our ability at present, however, to rigorously solve such problems in the realm of three-dimensional elasticity is limited to a few isolated results involving infinite fibers bonded along their entire length to fully-infinite matrices. Two noteworthy examples are the load-diffusion from an axially loaded fiber (Muki and Sternberg, 1969) and the load-absorption by a broken fiber in a remotely stressed medium (Ford, 1973). A much more interesting class of fiber–matrix system involves fibers embedded in semi-infinite half-spaces. This type of problem is pertinent to the study of how fiber-bridging in the wake of crack advance serves to isolate the crack tip from applied far-field loadings and thus increases the fracture toughness of a material. The analytic complexity of such problems, however, discourages any attempt at a rigorous solution. What is being proposed in this paper is a model for approximating load-diffusion in these systems.

One such model has already been developed by Muki and Sternberg (1970) and used to study such problems as load-transfer to a half-space from a partially embedded axially loaded rod and load-absorption by a semi-infinite fiber in a remotely stressed, fully-infinite matrix (Sternberg, 1970). Muki and Sternberg's model replaces the fiber–matrix system of the problem with an extended matrix occupying the volume originally containing both the fiber and the matrix and possessing the same elastic properties as the original matrix. This extended matrix is in turn reinforced by a "fictitious stiffener" whose modulus of elasticity when taken in sum with that of the extended matrix is equal to that of the original fiber. This stiffener is taken to be a one-dimensional elastic continuum bonded to the extended matrix in such a way that the axial strain in the stiffener is equal to the average extensional strain of the extended matrix in the volume occupied by and in the direction of the original fiber. Poisson's effect in the stiffener, and therefore in the fiber, is not taken into account. Finally, "bond-forces" are regarded as body forces uniformly distributed over disks perpendicular to the axis of the fiber and the load carried by the original fiber is equated with the sum of the stiffener load and the resultant load carried by the extended matrix in the bonded region.

A variation of Muki and Sternberg's model was used by Pak (1989) in a study of flexure of partially embedded fibers under lateral loads. The concept of a "fictitious stiffener" replacing the original fiber and treated as a one-dimensional elastic continuum was again employed. In this case, however, lateral displacement of the stiffener was taken to equal lateral displacement in the extended matrix along the centroidal axis of the original fiber and Bernoulli–Euler bending beam theory was used to describe the behavior of the stiffener. Body-force field distributions corresponding to laterally-loaded rigid disks embedded in the matrix along the axis of the fiber were adopted as the "bond-forces".

In the model proposed here the effect of the fiber on the matrix is assumed to be approximated by unknown distributions of axial forces and dilatations in an elastic space along the line where the fiber axis would lie. Mathematically the elastic field in the matrix is represented in terms of integrals with kernel functions corresponding to concentrated loads and dilatations. The fiber is modeled by a one-dimensional rod theory in which Poisson expansions and contractions are allowed. The two unknown distributions are determined by enforcing fiber equilibrium and continuity of tractions and displacements at the fiber–matrix interface leading mathematically to a pair of coupled integral equations. This model would seem to be conceptually "clean"; however, there is a difficulty. A concentrated axial force applied to the (model) fiber necessarily produces a discontinuity in axial strain. On the other hand, any distribution whatever of axial forces and dilatations according to the model produces continuous axial strains in the matrix at the fiber–matrix interface. This fundamental inconsistency is avoided by introducing an approximate expression for the axial strain in the matrix which has the proper discontinuity but which differs from the exact expression over a distance the order of a fiber radius. It is difficult to give a rigorous assessment of the errors involved in the approximate theory but it is thought to be accurate except within distances the order of a fiber radius from concentrated loads or other discontinuities. In contrast to the model used by Muki and Sternberg, this approach treats the transfer of load between the fiber and matrix in a manner which explicitly includes tangential tractions across the interface and therefore affords one more flexibility in examining systems where interface conditions are an issue. Furthermore, the fundamental elastostatic solutions in application in this model are those for a point force and a point dilatation. These solutions are much less cumbersome than the disk of uniform loading (or laterally-loaded rigid disk) required in Muki and Sternberg's model. These factors make the method presented below attractive for modeling a variety of fiber-matrix systems of interest to those studying fiber-reinforced materials.

2. LOAD-TRANSFER TO AN ELASTIC MEDIUM FROM AN INFINITE AXIALLY LOADED FIBER

Perhaps the best way to present this model is to demonstrate its application with a simple problem, in this case the three-dimensional analog of Melan's first problem from two-dimensional elasticity. An infinite cylindrical fiber, with a circular cross-section of radius a, is ideally bonded along its entire length to a fully-infinite matrix and subjected to a concentrated load F (see Fig. 1). The model is used to solve for the resultant axial load carried by the fiber. A cylindrical coordinate system is defined as shown in Fig. 1 with the z-axis coincident with the centroidal axis of the fiber and the applied load at the origin in the negative z-direction. Both the fiber and the matrix are homogeneous and isotropic, linear elastic solids with Young's modulus and Poisson's ratio taken respectively to be E_f and ν_f for the fiber and E_m and ν_m for the matrix.

Consider first the fiber of the problem. In this model the fiber is approximated as an axisymmetric elastic rod with a uniform axial stress σ. This means it is assumed that $\varepsilon_\theta = \varepsilon_r$, ε_z, $\sigma_\theta = \sigma_r$, and σ are functions of z only and shear strains are ignored. Under the rod theory approximation, constitutive relations for the fiber reduce to

$$\sigma = E_f \varepsilon_z + 2\nu_f \sigma_r \tag{1}$$

and

$$E_f \varepsilon_\theta + \nu_f E_f \varepsilon_z - (1-2\nu_f)(1+\nu_f)\sigma_r = 0. \tag{2}$$

The fiber, taken as a free body, is subject to a concentrated load F at $z = 0$ and to bonding tractions acting at $r = a$ between the fiber and the matrix. These bonding tractions, along with their equivalent matrix stresses, are a distributed shear stress, $\tau = \tau_{rz}^m(a, z)$, and a self-equilibrating "pressure", $\sigma_r = \sigma_r^m(a, z)$. Throughout the remainder of this paper, field

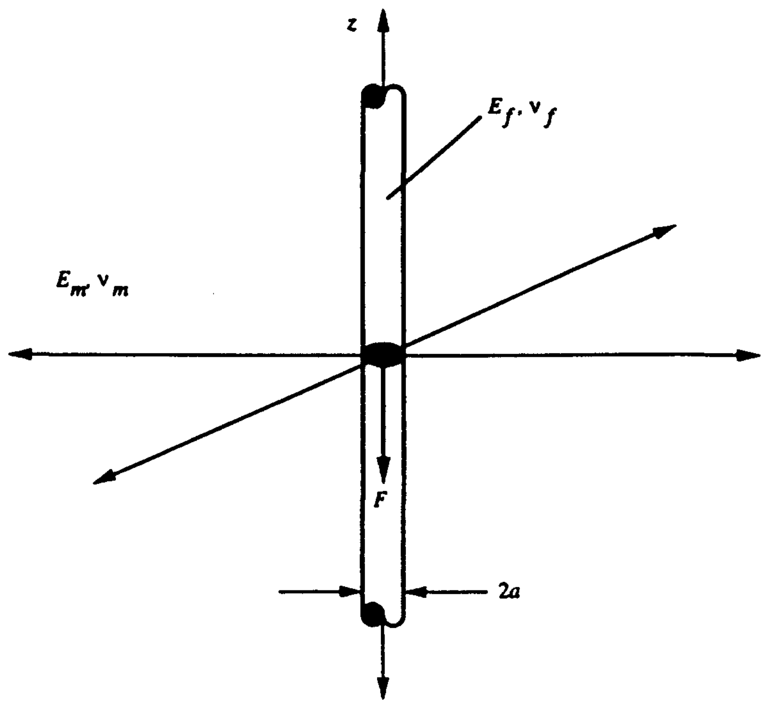

Fig. 1. Infinite cylindrical fiber embedded in an infinite elastic matrix.

quantities in the matrix will be denoted by superscribed *m*s. The rod is in equilibrium if, for all z,

$$\pi a^2\sigma + 2\pi a \int_0^z \tau \, dz' = \frac{F}{2}\operatorname{sgn}(z) \tag{3}$$

where the signum function $\operatorname{sgn}(z) \equiv z/|z|$. This equation might of course appear in different forms depending on the lower limit used in the integral of the shear stress distribution. As written here, however, (3) reflects the natural symmetry of the problem. In what follows, (2) and (3) are taken to be the governing equations. Utilizing (1), the governing equations contain three fiber quantities which need to be related to the approximate elastic field in the matrix described below, namely ε_θ, ε_z, and σ_r.

As already stated, the elastic field in the matrix acted upon by the loaded fiber is approximated by the reaction in a fully-infinite elastic space due to a concentrated force F acting at the origin in the negative z-direction along with distributions along the z-axis of point forces and point dilatations, $p(z)$ and $q(z)$ respectively, where the point force distribution must be self-equilibrating. Papkovich stress functions, ψ and ϕ, will be used to express this approximate elastic field. In a cylindrical coordinate system with rotational symmetry the expressions for radial and axial displacements are

$$u(r,z) = \frac{1+\nu}{E}\frac{\partial}{\partial r}[z\psi(r,z) + \phi(r,z)] \tag{4}$$

and

$$w(r,z) = \frac{1+\nu}{E}\left[z\frac{\partial}{\partial z}\psi(r,z) - (3-4\nu)\psi(r,z) + \frac{\partial}{\partial z}\phi(r,z)\right]. \tag{5}$$

Thus, using the Kelvin solution for a concentrated load in an infinite elastic space along with that for a point dilatation (Sokolnikoff, 1956), the approximate elastic field can be expressed as

$$\psi(r,z) = \frac{1}{8\pi(1-\nu_m)}\int_{-\infty}^{\infty}\frac{1}{\rho(r,z-\zeta)}[F\delta(\zeta)+p(\zeta)]\,\mathrm{d}\zeta$$

$$\phi(r,z) = \frac{-1}{8\pi(1-\nu_m)}\int_{-\infty}^{\infty}\frac{\zeta}{\rho(r,z-\zeta)}p(\zeta)\,\mathrm{d}\zeta-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\rho(r,z-\zeta)}q(\zeta)\,\mathrm{d}\zeta \tag{6}$$

where $\rho(r,z)\equiv\sqrt{r^2+z^2}$ and $\delta(\zeta)$ is the Dirac function. In this expression the point forces are of strength $p(z)$ and act in the negative z-direction, while the point dilatations in expansion are of magnitude $[(1+\nu_m)/E_m]q(z)$. Using these stress functions, expressions for all field variables in the matrix can be derived in the form of infinite integrals of the unknown distributions multiplied by some known difference kernel, e.g. in the form

$$\varepsilon_z^m(r,z) = \int_{-\infty}^{\infty}\{A_1(r,z-\zeta)[F\delta(\zeta)+p(\zeta)]+A_2(r,z-\zeta)q(\zeta)\}\,\mathrm{d}\zeta \tag{7}$$

where the kernel functions A_1 and A_2 are real analytic functions of z for $r>0$ and so, consequently, is ε_z^m. For those matrix quantities which shall be used below in terms of the Papkovich stress functions, see the Appendix. For reasons discussed in the literature (Muki and Sternberg, 1969) having to do with the singular nature of (7) in the limit as $r\to 0$ it is impossible to model the fiber as an elastic line in this problem.

The fiber quantities ε_θ and σ_r in eqns (2) and (3) are related to matrix quantities at $r=a$ by continuity of tractions and displacements, so that

$$\varepsilon_\theta = \frac{1}{a}u^m(a,z) \tag{8}$$

and

$$\sigma_r = \sigma_r^m(a,z) \tag{9}$$

where we have shown that each of these quantities is analytic and must, therefore, be continuous. On the other hand, on physical grounds, and from (3), the axial fiber stress σ must certainly be discontinuous. Then, assuming that σ_r is continuous, it follows from (1) that ε_z is discontinuous. The difficullty is that the seemingly most natural expression for ε_z, that given by

$$\varepsilon_z = \varepsilon_z^m(a,z) \tag{10}$$

in the form (7), cannot be discontinuous and so is suitable for use only in eqn (2). Some alternative expression for ε_z in terms of matrix field quantities must be adopted for eqn (3).

As an indication that some alternative expression for ε_z involving a discontinuity would not be unreasonable, consider the following. Let $\bar{\varepsilon}_z^*$ be the average, over a disk of radius a centered on the z-axis, of the strains in the matrix $\varepsilon_z^{*m}(r,z-\zeta)$ where quantities with a superscript asterisk are due to a unit concentrated axial load at $(r,z)=(0,\zeta)$. Calculation leads to an expression of the form

$$\bar{\varepsilon}_z^* = C\,\mathrm{sgn}\,(z-\zeta)+\text{regular terms}. \tag{11}$$

Except in the range $|z-\zeta| = O(a)$ this expression agrees rather closely with $\varepsilon_z^{*m}(a,z-\zeta)$ for the same load. However, this expression is not suitable for the following reason. From (1) it follows that the "jump" $\Delta\varepsilon_z$ in strain be related to the jump in σ_z at $z=\zeta$ by

$$\Delta\varepsilon_z = \frac{1}{E_f}\Delta\sigma_z = \frac{1}{\pi a^2 E_f}$$

(for the supposed unit load). The constant C in (11) does not meet this requirement. Instead, a form relating axial strain in the fiber to matrix quantities was adopted that combined continuity of displacements (10) with a term containing the proper discontinuity and decaying rapidly outside the range $|z-\zeta| = O(a)$. For a unit axial load at $(0, \zeta)$

$$\varepsilon_z^* = \frac{1}{2\pi a^2 E_f}\left[\operatorname{sgn}(z-\zeta) - \operatorname{erf}\left(\frac{z-\zeta}{a}\right)\right] + \varepsilon_z^{*m}(a, z-\zeta). \tag{12}$$

A somewhat different expression will be used in the half-space problem to follow. There is certainly nothing unique about the expression (12) which gives finally an expression for axial strain to be used in eqns (1) and (3). Final numerical results, which agree very well with those in the literature, seem to indicate that such results are quite insensitive to the precise definition of ε_z^*.

Using the expressions established above to relate the unknown fiber quantities to the elastic field given by (6), and non-dimensionalizing with a and F, the governing equations (2) and (3) can be rewritten as a pair of coupled integral equations in terms of the unknown distributions $p(z)$ and $q(z)$;

$$2\int_0^z p(\zeta)\,d\zeta + \int_{-\infty}^{\infty} [\Gamma_{11}(z-\zeta)p(\zeta) + \Gamma_{12}(z-\zeta)q(\zeta)]\,d\zeta = -\Gamma_{11}(z) \tag{13}$$

$$\int_{-\infty}^{\infty} [\Gamma_{21}(z-\zeta)p(\zeta) + \Gamma_{22}(z-\zeta)q(\zeta)]\,d\zeta = -\Gamma_{21}(z) \tag{14}$$

where the kernel functions $\Gamma_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are real analytic functions (see the Appendix). Note that this system could be solved analytically using Fourier transform methods and the convolution theorem. However, of concern here is the establishment of methodology for more complicated load-diffusion problems.

The system (13) and (14) is first reduced to a set of discrete linear equations. Using the symmetry of the distributions, $p(z)$ even and $q(z)$ odd, the infinite integrals can be rewritten in semi-infinite form, though without difference kernels. Truncating infinite limits at appropriately large values and approximating the integrals with a trapezoidal quadrature scheme, the two equations are then enforced at the discrete quadrature points in accordance with the Nystrom method (Delves and Mohamed, 1985). Though not truly a system of first kind integral equations, (13) and (14) unfortunately retain some of the ill-posed behavior inherent in all such equations. This is dealt with by using singular value decomposition to solve the set of linear equations, filtering out small length scale instabilities with some unavoidable degradation of the results for small z. Under the assumptions of this model, especially that the fiber behaves as an elastic rod and has axial strain given by (12), one would not expect high accuracy near the applied load in any event. The use of singular value decomposition in the solution of such problems is well understood and more detailed discussion can be found in the literature (Barakat and Buder, 1979; Press *et al.*, 1986). The necessary numerical routines for this method can be found in both the literature and in commercial software libraries.

Axial load in the fiber is determined by eqn (1) [or equivalently eqn (3)]. Rewriting (1) in non-dimensional form with the now known distributions $p(z)$ and $q(z)$ gives

$$\frac{\sigma(z)}{\sigma_0} = \Lambda_1(z) + \int_{-\infty}^{\infty} [\Lambda_1(z-\zeta)p(\zeta) + \Lambda_2(z-\zeta)q(\zeta)]\,d\zeta \tag{15}$$

where $\sigma_0 = F/2\pi a^2$, Λ_1 is discontinuous at $z = 0$, and Λ_2 is a real analytic function (see the

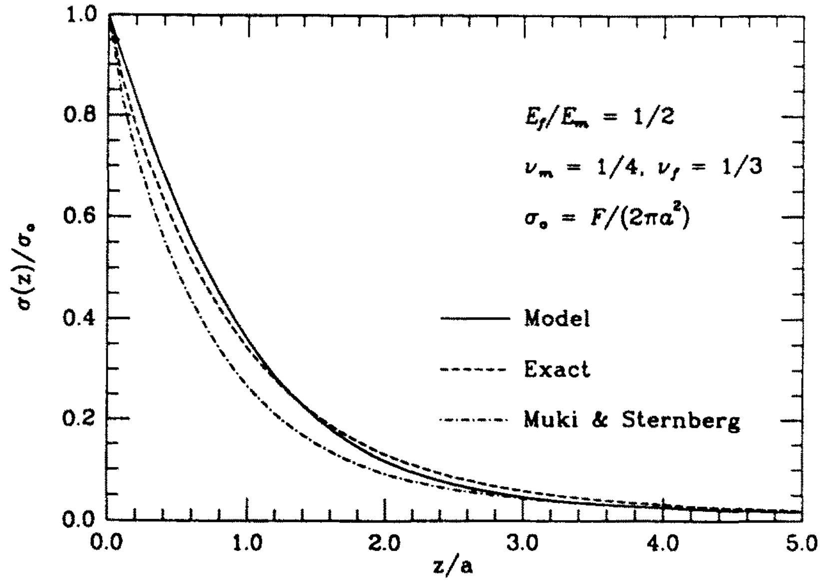

Fig. 2. Load-diffusion results in the three-dimensional analog of Melan's first problem for the model presented in this paper and Muki and Sternberg's exact and approximate formulations.

Appendix). The bonding tractions across the fiber–matrix interface, τ and σ_r, can similarly be determined. Comparison between the results obtained with this model and those from an exact elastostatic solution developed by Muki and Sternberg (1969), along with the results of their approximate model, are shown in Figs 2–4. The results for σ very closely approximate the exact solution and can be shown to have the same asymptotic form (Muki and Sternberg, 1969) in the highest order term as $|z| \to \infty$, i.e.

$$\frac{\sigma(z)}{\sigma_0} = (1+\nu_m)\frac{E_f}{E_m}\left[1 - \frac{\nu_f(1-2\nu_f)}{(1+\nu_m)E_f/E_m + (1+\nu_f)(1-2\nu_f)}\right]\frac{\mathrm{sgn}\,(z)}{z^2} + o(z^{-2}). \tag{16}$$

The ratio of Young's moduli between the fiber and matrix is seen to be much more of an influential factor than either of the Poisson's ratios. A comparison of σ_r in Fig. 3 points to

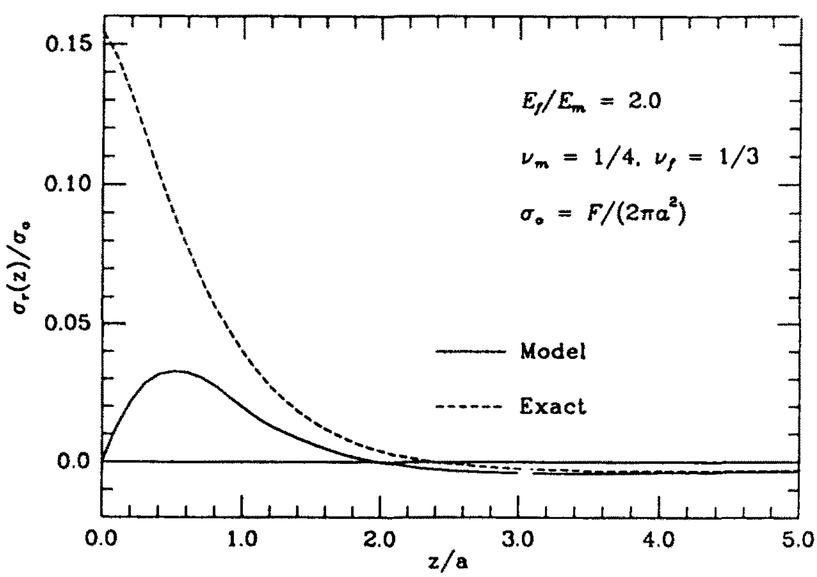

Fig. 3. Normal stress across the fiber–matrix interface in the three-dimensional analog of Melan's first problem for the model presented in this paper and the exact solution.

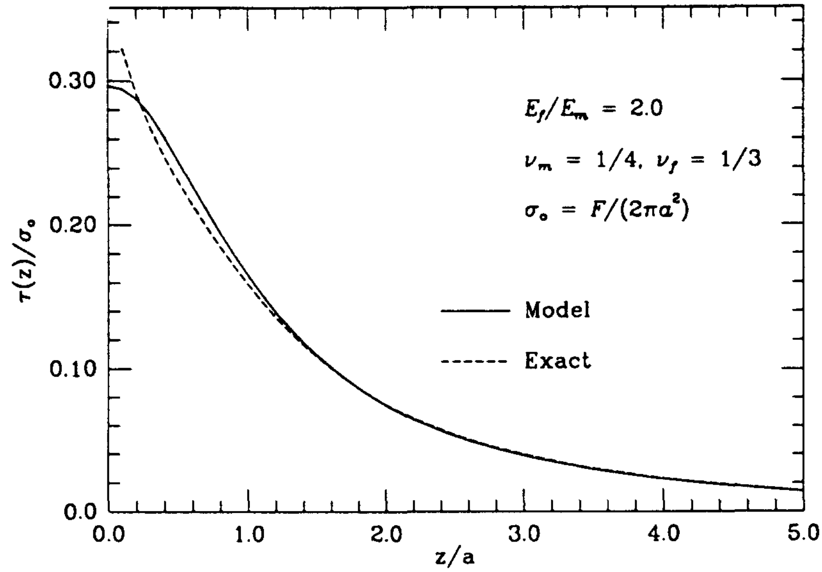

Fig. 4. Shear stress across the fiber–matrix interface in the three-dimensional analog of Melan's first problem for the model presented in this paper and the exact solution.

a shortcoming of the model which gives a result that is incorrectly continuous, though it approaches the exact solution asymptotically for large z. However, consider for a moment the exact solution for the problem in which a point force acts at the centroidal axis of the fiber. In that case all field variables (σ, in particular) would be continuous everywhere on the fiber-matrix interface, more in agreement with the present model. In any case, the model is an approximation and cannot be expected to predict accurately details near concentrated loads or other singularities. Figure 4 shows that while the exact solution predicts a logarithmic singularity in τ at $z = 0$, the everywhere bounded approximate solution is quite accurate elsewhere. Any approximate model (including that of Muki and Sternberg) is most likely to be inaccurate near singularities. This does not vitiate their usefulness in fracture mechanics applications where path (or surface) independent integrals play a decisive role.

3. LOAD-TRANSFER TO AN ELASTIC HALF-SPACE FROM A SEMI-INFINITE AXIALLY LOADED FIBER

Typically of greater interest in the study of fiber-reinforced materials are problems of load-diffusion from fibers in semi-infinite half-spaces. The problem of this type solved here is the three-dimensional analog of Reissner's problem from two-dimensional elasticity. A semi-infinite cylindrical fiber, with a circular cross-section of radius a, is ideally bonded to a semi-infinite matrix. The fiber is normal to the free-surface of the matrix and is subjected to a concentrated load F away from the matrix (see Fig. 5). A cylindrical coordinate system is defined as shown in Fig. 5 with the z-axis coincident with the centroidal axis of the fiber and the matrix occupying the space $z > 0$. Both the fiber and the matrix are homogeneous and isotropic, linear elastic solids with Young's modulus and Poisson's ratio taken respectively to be E_f and ν_f for the fiber and E_m and ν_m for the matrix.

In applying the model to this problem the procedure established above is repeated. Constitutive relations for the fiber under the rod theory approximation are still given by (1) and (2) and the bonding tractions along the fiber, with their equivalent matrix stresses, are denoted in the same way. The rod is in equilibrium if, for all $z \geqslant 0$,

$$\pi a^2\sigma + 2\pi a \int_0^z \tau \, dz' = F. \tag{17}$$

In what follows, (2) and (17) are taken to be the governing equations and ε_θ, ε_z, and σ_r are

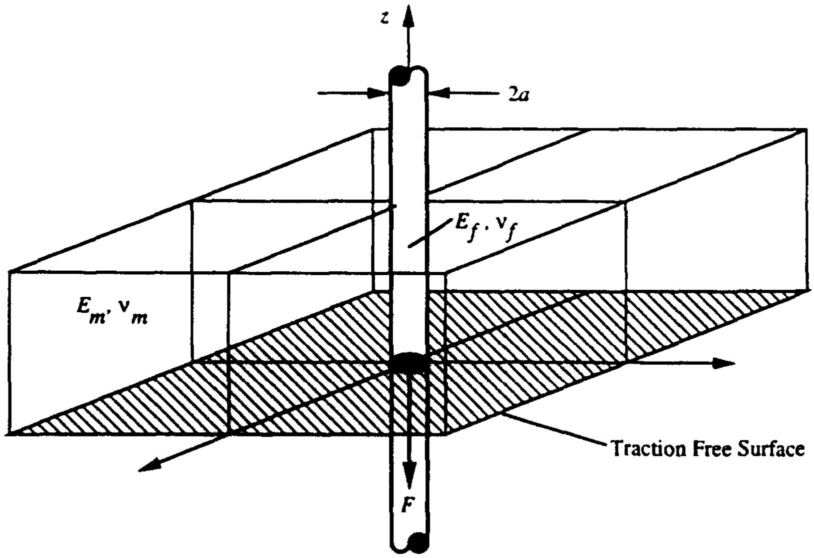

Fig. 5. Semi-infinite cylindrical fiber embedded in a semi-infinite elastic matrix.

once again the fiber quantities which need to be related to the approximate elastic field in the matrix described below.

The approximate elastic field due to a concentrated force F acting at the origin in the negative z-direction along with distributions along the positive z-axis of point forces and point dilatations, $p(z)$ and $q(z)$ respectively, can be expressed using the Mindlin solution for a concentrated load in a semi-infinite elastic half-space (Mindlin, 1953), along with that for a point dilatation (Sokolnikoff, 1956), as

$$\psi(r,z) = \frac{1}{8\pi(1-\nu_m)}\int_0^\infty \left\{\frac{2\zeta(z+\zeta)}{\rho^3(r,z+\zeta)} + \frac{3-4\nu_m}{\rho(r,z+\zeta)} + \frac{1}{\rho(r,z-\zeta)}\right\}[F\delta(\zeta)+p(\zeta)]\,\mathrm{d}\zeta + \frac{1}{2\pi}\int_0^\infty \frac{z+\zeta}{\rho^3(r,z+\zeta)}q(\zeta)\,\mathrm{d}\zeta$$

$$\phi(r,z) = \frac{-1}{8\pi(1-\nu_m)}\int_0^\infty \left\{\frac{(3-4\nu_m)\zeta}{\rho(r,z+\zeta)} - 4(1-\nu_m)(1-2\nu_m)\log[z+\zeta+\rho(r,z+\zeta)] + \frac{\zeta}{\rho(r,z-\zeta)}\right\}[F\delta(\zeta)+p(\zeta)]\,\mathrm{d}\zeta - \frac{1}{4\pi}\int_0^\infty \left\{\frac{3-4\nu_m}{\rho(r,z+\zeta)} + \frac{1}{\rho(r,z-\zeta)}\right\}q(\zeta)\,\mathrm{d}\zeta \tag{18}$$

where it is recalled that $\rho(r,z) \equiv \sqrt{r^2+z^2}$. The distributions have the same magnitude as before. Using these stress functions results in matrix field variables of the form

$$\varepsilon_z^m(r,z) = \int_0^\infty \{B_1(r,z,\zeta)[F\delta(\zeta)+p(\zeta)] + B_2(r,z,\zeta)q(\zeta)\}\,\mathrm{d}\zeta. \tag{19}$$

While the kernels B_1 and B_2 are not difference kernels they are still real analytic functions of z and ζ for $r > 0$ and so, consequently, is ε_z^m.

The fiber–matrix relations for ε_θ and σ_r that were established in (8) and (9) are still valid as is that for axial strain, ε_z, that was established in (10) to be used in eqn (2). Care is used in choosing an expression for ε_z to be used in (17). Consider a unit concentrated axial load at $(0,\zeta)$. It follows from fiber equilibrium and (1) that there must be a jump in axial strain, $\Delta\varepsilon_z^* = 1/\pi a^2 E_f$, across the load. Recall that a superscribed asterisk denotes a quantity due to a unit concentrated axial load. The expression given in (12) satisfies this condition, but following the asymmetric nature of the problem the effective axial strain in the matrix was instead taken to be

$$\varepsilon_z^* = \frac{1}{\pi a^2 E_f} H(z-\zeta)\left[1-\operatorname{erf}\left(\frac{z-\zeta}{a}\right)\right]+\varepsilon_z^{*m}(a,z,\zeta) \tag{20}$$

where $H(z)$ is the Heaviside step function. Comparison with other results in the literature shows (20) to be an acceptable choice.

Using the expressions established above to relate the unknown fiber quantities to the elastic field given by (18), and non-dimensionalizing with a and F, the governing equations (2) and (17) can be written as a pair of coupled integral equations in terms of the unknown distributions $p(z)$ and $q(z)$:

$$\int_0^z p(\zeta)\,\mathrm{d}\zeta+\int_0^\infty [\Pi_{11}(z,\zeta)p(\zeta)+\Pi_{12}(z,\zeta)q(\zeta)]\,\mathrm{d}\zeta = -\Pi_{11}(z,0) \tag{21}$$

$$\int_0^\infty [\Pi_{21}(z,\zeta)p(\zeta)+\Pi_{22}(z,\zeta)q(\zeta)]\,\mathrm{d}\zeta = -\Pi_{21}(z,0) \tag{22}$$

where the kernel functions $\Pi_{\alpha\beta}$ $(\alpha,\beta = 1,2)$ are real analytic functions (see the Appendix). These equations are solved for discrete values of the distributions in the same way as before except that the limits of integration are already semi-infinite. Note that, as previously alluded to, (21) and (22) do not have difference kernels and could not, therefore, be solved using transform methods.

Axial load in the fiber is determined by eqn (1) [or equivalently eqn (17)]. Rewriting (1) in non-dimensional form with the now known distributions $p(z)$ and $q(z)$ gives

$$\frac{\sigma(z)}{\sigma_0} = \Sigma_1(z,0)+\int_0^\infty [\Sigma_1(z,\zeta)p(\zeta)+\Sigma_2(z,\zeta)q(\zeta)]\,\mathrm{d}\zeta \tag{23}$$

where $\sigma_0 = F/\pi a^2$, $\Sigma_1(z,\zeta)$ is discontinuous at $z = \zeta$, and Σ_2 is a real analytic function (see the Appendix). Results from (23) are compared with those from Muki and Sternberg's model (1970) and shown in Fig. 6.

4. CONCLUDING REMARKS

Comparison with an exact solution for the three-dimensional analog of Melan's first problem (Muki and Sternberg, 1969) shows that the fiber load-diffusion model gives good results for both axial load in the fiber and tangential tractions on the fiber–matrix interface at distances from the applied load greater than approximately one fiber radius. The discontinuous nature of the normal interface tractions, however, is not adequately accounted for. To demonstrate the model's application to fiber load-diffusion problems involving half-spaces the three-dimensional analog to Reissner's problem was examined. The results were compared to those from Muki and Sternberg's approximation (1970) and found to be in agreement.

It is hoped that the model will prove useful in the study of other more complex fiber–matrix systems which more closely resemble those observed in actual fiber-reinforced materials. In particular it is anticipated that the model will be useful in studying systems with prescribed interface conditions other than ideal bonding. Presently problems involving fiber debonding, frictional sliding, etc. are handled with evident success by a somewhat crude "shear lag" model (Hutchinson and Jensen, 1990). Perhaps with some further development the present model can be used to validate the far simpler shear lag model.

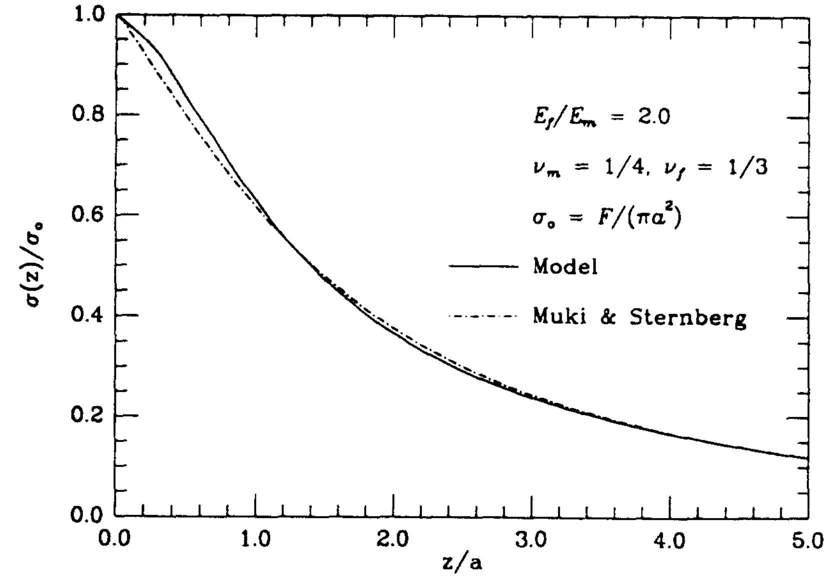

Fig. 6. Load-diffusion results in the three-dimensional analog of Reissner's problem for the model presented in this paper and Muki and Sternberg's approximate solution.

Acknowledgements—This work was supported in part by the DARPA University Research Initiative (Subagreement P.O. VB38639-0 with the University of California, Santa Barbara, ONR Prime Contract N00014-86-K-0753), the Office of Naval Research (Contract N00014-90-J-1377), and the Division of Applied Sciences, Harvard University.

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APPENDIX

Listed below are those matrix quantities that are needed for the implementation of the model in this paper. They are given in terms of the Papkovich stress functions which in turn are given, in terms of the unknown distributions $p(z)$ and $q(z)$, by eqn (6) for the full-space problem and eqn (18) for the half-space problem:

$$\frac{1}{r}u_r^m(r,z) = \frac{1+\nu_m}{E_m}\frac{\partial}{\partial r}[r\psi(r,z)+\phi(r,z)] \tag{A1}$$

$$\varepsilon_z^m(r,z) = \frac{1+\nu_m}{E_m}\left[z\frac{\partial^2}{\partial z^2}\psi(r,z)-2(1-2\nu_m)\frac{\partial}{\partial z}\psi(r,z)+\frac{\partial^2}{\partial z^2}\phi(r,z)\right] \tag{A2}$$

$$\sigma_r^m(r,z) = \frac{\partial^2}{\partial r^2}[z\psi(r,z)+\phi(r,z)]-2\nu_m\frac{\partial}{\partial z}\psi(r,z) \tag{A3}$$

$$\int_0^z \tau_{rz}^m(r,z')\,\mathrm{d}z' = \frac{\partial}{\partial r}[z'\psi(r,z')+\phi(r,z')]\Big|_{z'=0}^{z'=z} - 2(1-\nu_m)\int_0^z \frac{\partial}{\partial r}\psi(r,z')\,\mathrm{d}z'. \tag{A4}$$

Note that, as stated in the text, each of these quantities is a real analytic function of z for $r > 0$.

The three-dimensional analog of the Melan problem

The kernel functions for the pair of coupled integral equations (13) and (14) are arrived at by substituting (A1)–(A4) [with ψ and ϕ given by (6)] into the governing equations (2) and (3) using the fiber–matrix relations established in the body of the paper. To non-dimensionalize with a and F the following substitutions are made: $z \to az$, $\zeta \to a\zeta$, $\rho_a \to a\rho$, $\delta(\zeta) \to (1/a)\delta(\zeta)$, $p(\zeta) \to (F/a)p(\zeta)$, and $q(\zeta) \to Fq(\zeta)$. Letting $\rho_z \equiv \sqrt{1+z^2}$, the kernel functions are

$$\Gamma_{11}(z) = -\operatorname{erf}(z) + \frac{z}{\rho_z} + \alpha_1\frac{z}{\rho_z^3} + \alpha_2\frac{z}{\rho_z^5} \tag{A5}$$

$$\Gamma_{12}(z) = \alpha_3\frac{1}{\rho_z^3} + \alpha_4\frac{1}{\rho_z^5} \tag{A6}$$

$$\Gamma_{21}(z) = \alpha_5\frac{z}{\rho_z^3} + \alpha_6\frac{z}{\rho_z^5} \tag{A7}$$

$$\Gamma_{22}(z) = \alpha_7\frac{1}{\rho_z^3} + \alpha_8\frac{1}{\rho_z^5} \tag{A8}$$

where

$$\alpha_1 = (1+\nu_m)\frac{E_f}{E_m} - \frac{1+\nu_f(1-2\nu_m)}{2(1-\nu_m)} \tag{A9}$$

$$\alpha_2 = \frac{3}{2(1-\nu_m)}\left[\nu_f - \tfrac{1}{2}(1+\nu_m)\frac{E_f}{E_m}\right] \tag{A10}$$

$$\alpha_3 = 1+\nu_f-(1+\nu_m)\frac{E_f}{E_m} \tag{A11}$$

$$\alpha_4 = \tfrac{3}{2}(1+\nu_m)\frac{E_f}{E_m} - 3\nu_f \tag{A12}$$

$$\alpha_5 = (1+\nu_m)\left[\frac{1}{2(1-\nu_m)} - 2\nu_f\right]\frac{E_f}{E_m} - \frac{(1-2\nu_f)(1+\nu_f)(1-2\nu_m)}{2(1-\nu_m)} \tag{A13}$$

$$\alpha_6 = \frac{3\nu_f}{2}\left(\frac{1+\nu_m}{1-\nu_m}\right)\frac{E_f}{E_m} + \frac{3(1-2\nu_f)(1+\nu_f)}{2(1-\nu_m)} \tag{A14}$$

$$\alpha_7 = (1-2\nu_f)\left[1+\nu_f-(1+\nu_m)\frac{E_f}{E_m}\right] \tag{A15}$$

$$\alpha_8 = -3\nu_f(1+\nu_m)\frac{E_f}{E_m} - 3(1-2\nu_f)(1+\nu_f). \tag{A16}$$

The kernel functions in the expression for axial load in the fiber (15) are

$$\Lambda_1(z) = \operatorname{sgn}(z) - \operatorname{erf}(z) + \beta_1\frac{z}{\rho_z^3} + \beta_2\frac{z}{\rho_z^5} \tag{A17}$$

$$\Lambda_2(z) = \beta_3\frac{1}{\rho_z^3} + \beta_4\frac{1}{\rho_z^5} \tag{A18}$$

where

$$\beta_1 = (1+\nu_m)\frac{E_f}{E_m} - \frac{\nu_f(1-2\nu_m)}{2(1-\nu_m)} \tag{A19}$$

$$\beta_2 = \alpha_2 = \frac{3}{2(1-\nu_m)}\left[\nu_f - \tfrac{1}{2}(1+\nu_m)\frac{E_f}{E_m}\right] \tag{A20}$$

$$\beta_3 = \nu_f - (1+\nu_m)\frac{E_f}{E_m} \tag{A21}$$

$$\beta_4 = \alpha_4 = \tfrac{3}{2}(1+\nu_m)\frac{E_f}{E_m} - 3\nu_f. \tag{A22}$$

The three-dimensional analog of Reissner's problem

The kernel functions for the pair of coupled integral equations (21) and (22) are arrived at by substituting (A1)–(A4) [with ψ and ϕ given by (18)] into the governing equations (2) and (17) using the fiber–matrix relations established in the body of the paper. The same substutitions are made to non-dimensionalize with a and F. The kernel functions are

$$\Pi_{11}(z,\zeta) = -H(z-\zeta)\,\mathrm{erf}\,(z-\zeta) + \tfrac{1}{8}\left(\frac{1+\nu_m}{1-\nu_m}\right)\frac{E_f}{E_m}\Omega_1(z,\zeta) + \frac{\nu_f}{4(1-\nu_m)}\Omega_2(z,\zeta) + \frac{1}{4(1-\nu_m)}\Omega_3(z,\zeta) \tag{A23}$$

$$\Pi_{12}(z,\zeta) = \tfrac{1}{4}(1+\nu_m)\frac{E_f}{E_m}\Omega_4(z,\zeta) + \frac{\nu_f}{2}\Omega_5(z,\zeta) + \tfrac{1}{2}\Omega_6(z,\zeta) \tag{A24}$$

$$\Pi_{21}(z,\zeta) = \frac{\nu_f}{8}\left(\frac{1+\nu_m}{1-\nu_m}\right)\frac{E_f}{E_m}\Omega_1(z,\zeta) - \frac{(1+\nu_f)(1-2\nu_f)}{8(1-\nu_m)}\Omega_2(z,\zeta) + \tfrac{1}{8}\left(\frac{1+\nu_m}{1-\nu_m}\right)\frac{E_f}{E_m}\Omega_7(z,\zeta) \tag{A25}$$

$$\Pi_{22}(\zeta) = \frac{\nu_f}{4}(1+\nu_m)\frac{E_f}{E_m}\Omega_4(z,\zeta) - \tfrac{1}{4}(1+\nu_f)(1-2\nu_f)\Omega_5(z,\zeta) + \tfrac{1}{4}(1+\nu_m)\frac{E_f}{E_m}\Omega_8(z,\zeta) \tag{A26}$$

where $\Omega_i(z,\zeta)$ $(i = 1,\ldots,8)$ are continuous, analytic functions given below. Letting $\rho \equiv \sqrt{1+(z-\zeta)^2}$ and $\bar{\rho} \equiv \sqrt{1+(z+\zeta)^2}$,

$$\Omega_1(z,\zeta) = \frac{-30z\zeta(z+\zeta)}{\bar{\rho}^7} - \frac{3(1-4\nu_m-4z\zeta)(z+\zeta)+6z}{\bar{\rho}^5} + \frac{4(1-4\nu_m+2\nu_m^2)(z+\zeta)+4z}{\bar{\rho}^3} - \frac{3(z-\zeta)}{\rho^5} + \frac{4(1-\nu_m)(z-\zeta)}{\rho^3} \tag{A27}$$

$$\Omega_2(z,\zeta) = \frac{30z\zeta(z+\zeta)}{\bar{\rho}^7} - \frac{6z\zeta(z+\zeta)+12\nu_m z-9(z-\zeta)}{\bar{\rho}^5} + \frac{6\zeta-(1-2\nu_m)(3-4\nu_m)(z+\zeta)}{\bar{\rho}^3} + \frac{3(z-\zeta)}{\rho^5} - \frac{(1-2\nu_m)(z-\zeta)}{\rho^3} + 4(1-\nu_m)(1-2\nu_m)\frac{(z+\zeta)^3+\zeta(2+\zeta)\bar{\rho}}{(z+\zeta+\bar{\rho})^2\bar{\rho}^3} \tag{A28}$$

$$\Omega_3(z,\zeta) = \frac{-6z\zeta(z+\zeta)}{\bar{\rho}^5} - \frac{(3-4\nu_m)z+\zeta}{\bar{\rho}^3} - \frac{z-\zeta}{\rho^3} - \frac{2(1-\nu_m)}{(z+\zeta+\bar{\rho})\bar{\rho}} - \frac{2(1-\nu_m)}{(z-\zeta+\rho)\rho} + 4(1-\nu_m) \tag{A29}$$

$$\Omega_4(z,\zeta) = \frac{-30z(z+\zeta)}{\bar{\rho}^7} + \frac{12z(z+\zeta)-3+12\nu_m}{\bar{\rho}^5} + \frac{2-8\nu_m}{\bar{\rho}^3} + \frac{3}{\rho^5} - \frac{2}{\rho^3} \tag{A30}$$

$$\Omega_5(z,\zeta) = \frac{30z(z+\zeta)}{\bar{\rho}^7} - \frac{6z(z+\zeta)+9}{\bar{\rho}^5} + \frac{3+4\nu_m}{\bar{\rho}^3} - \frac{3}{\rho^5} + \frac{1}{\rho^3} \tag{A31}$$

$$\Omega_6(z,\zeta) = \frac{-6z(z+\zeta)}{\bar{\rho}^5} - \frac{1}{\bar{\rho}^3} + \frac{1}{\rho^3} \tag{A32}$$

$$\Omega_7(z,\zeta) = \frac{-6z\zeta(z+\zeta)}{\bar{\rho}^5} - \frac{(3-4\nu_m)(z-\zeta)}{\bar{\rho}^3} - \frac{z-\zeta}{\rho^3} + \frac{4(1-\nu_m)(1-2\nu_m)}{(z+\zeta+\bar{\rho})\bar{\rho}} \tag{A33}$$

$$\Omega_8(z,\zeta) = \frac{-6z(z+\zeta)}{\bar{\rho}^5} + \frac{3-4\nu_m}{\bar{\rho}^3} + \frac{1}{\rho^3}. \tag{A34}$$

The kernel functions in the expression for axial load in the fiber (23) are

$$\Sigma_1(z,\zeta) = H(z-\zeta)[1-\mathrm{erf}\,(z-\zeta)] + \frac{1}{8}\left(\frac{1+\nu_m}{1-\nu_m}\right)\frac{E_f}{E_m}\Omega_1(z,\zeta) + \frac{\nu_f}{4(1-\nu_m)}\Omega_2(z,\zeta) \tag{A35}$$

$$\Sigma_2(z,\zeta) = \tfrac{1}{4}(1+\nu_m)\frac{E_f}{E_m}\Omega_4(z,\zeta) + \frac{\nu_f}{2}\Omega_5(z,\zeta) \tag{A36}$$

where $\Omega_i(z,\zeta)$ $(i = 1,2,4,5)$ are as defined above.